Constrained Integer Partitions

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Abstract

We consider the problem of partitioning n integers into two subsets of given cardinalities such that the discrepancy, the absolute value of the difference of their sums, is minimized. The integers are i.i.d. random variables chosen uniformly from the set $\{1, \ldots, M\}$. We study how the typical behavior of the optimal partition depends on n, M and the bias s, the difference between the cardinalities of the two subsets in the partition. In particular, we rigorously establish this typical behavior as a function of the two parameters $\kappa := n^{-1} \log_2 M$ and $b := |s|/n$ by proving the existence of three distinct "phases" in the κb-plane, characterized by the value of the discrepancy and the number of optimal solutions: a "perfect phase" with exponentially many optimal solutions with discrepancy 0 or 1; a "hard phase" with minimal discrepancy of order $Me^{-\Theta(n)}$; and a "sorted phase" with an unique optimal partition of order Mn, obtained by putting the $(s + n)/2$ smallest integers in one subset.

1 Introduction

Phase transitions in random combinatorial problems have been the subject of much recent attention. The random optimum partitioning problem is the only NP-hard problem for which the existence of a sharp phase transition has been rigorously established, as have many detailed properties of the transition ([2], see [3] for a short overview). Here we study a constrained version of the random optimum partitioning problem, and extend some of the results of [2] to that case. Complete proofs of the results announced here will be given in [1].

The integer optimum partitioning problem is a classic problem of combinatorial optimization in which a given set of n integers is partitioned into two subsets in order to minimize the absolute value of the difference between the

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sum of the integers in the two subsets, the so-called *discrepancy.* Notice that for any given set of integers, the discrepancies of all partitions have the same parity, namely that of the sum of the n integers. We call a partition *perfect* if its discrepancy is 0, when this sum is even, or 1, when this sum is odd. The decision question is whether there exists a perfect partition. In the uniformly random version, an instance is a given a set of n i.i.d. integers drawn uniformly at random from $\{1, 2, \dots, M\}$. We will sometimes use the notation $m = \log_2 M$; notice that each of the random integers has m binary bits. Previous work had established a sharp transition as a function of the parameter $\kappa := m/n$, characterized by a dramatic change in the probability of a perfect partition. For M and n tending to infinity in the limiting ratio $\kappa = m/n$, the probability of a perfect partition tends to 0 for $\kappa < 1$, while the probability tends to 1 for $\kappa > 1$. This result was suggested by the work of one of the authors [13] and proved in a paper by the three other authors [2]. See [9] for a beautiful introduction to the optimum partitioning phase transition and some of its properties.

Here we consider a constrained variant of the problem in which we require that the two subsets have given cardinalities; we say that the difference of the two cardinalities is the *bias*, s, of the partition. We establish the phase diagram of the random constrained integer partitioning problem as a function of the two parameters $\kappa := m/n$ and $b := |s|/n$. In the language of statistical physics, b would be called the magnetization, and the problem considered here, where b is constrained to assume a particular value, would be called the "microcanonical" integer partitioning problem. Microcanonical random problems are known to be much more difficult than their unconstrained analogues.

There has been a good deal of rigorous and nonrigorous work on the random optimum partitioning problem in the theoretical computer science ([10], [4], [11], [12]), artificial intelligence ([8]), theoretical physics ([7]; [5], [6], [13], [14]) and mathematics ([3], [2]) communities. For our purposes here, the most relevant work is the rigorous analysis by three of the authors of this paper ([3], [2]), motivated by the theoretical statistical physics arguments by the other author of this paper [13]. Among other things, they established the existence of a transition at $\kappa_c = 1$ below which the probability of a perfect partition tends to one with n and m, and above which it tends to zero, and also gave the finite-size scaling window of the transition: namely, in terms of the more detailed parametrization $m = \kappa_n$ with $\kappa_n = 1 - \log_2 n/(2n) + \lambda_n/n$, the probability of a perfect partition tends to 1, 0, or a computable λ-dependent constant strictly between 0 and 1, depending on whether λ_n tends to $-\infty$, ∞, or $\lambda \in (-\infty, \infty)$, respectively. The analysis also determined the distribution of the number of perfect partitions, the distribution of the minimum discrepancy, and the joint distribution of the k smallest discrepancies, which give the entropy, the ground state energy and the bottom of the energy spectrum, respectively.

The location of the phase transition for the unconstrained problem immediately yields a one-dimensional phase diagram as a function of κ: For $\kappa \in (0, \kappa_c)$ with $\kappa_c = 1$, the system is in a "perfect phase" in which the probability of a perfect partition tends to 1 as M and n tend to infinity in the fixed function κ. For $\kappa \in (\kappa_c, \infty)$, the probability of a perfect partition tends to 0, and moreover, there is an unique optimal partition. We call this the "hard phase," since for $\kappa > \kappa_c$, it is presumably computationally difficult to find the optimal partition.

In this work, we consider the constrained optimum partitioning problem with bias s, as first proposed in [5], and extend the phase diagram to the two-dimensional κb-plane. See Figure 1. In addition to the extensions of the perfect and hard phases, we establish the existence of a new phase which we call the "sorted phase."

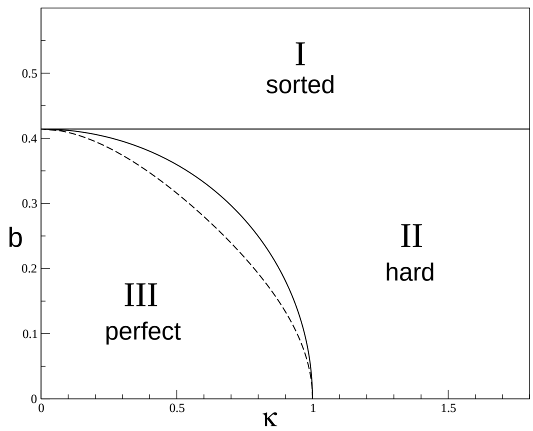

Fig. 1. Phase diagram of the constrained integer partitioning problem.

The sorted phase is easy to understand. One way to meet the bias constraint is to take the $(s + n)/2$ smallest integers and put them in one subset of the partition.[1] It is not difficult to see that the resulting "sorted partition" is optimal if the total weight of this subset is at least half of the sum of all n integers. We define the sorted phase as the subset of the κb-plane where the sorted partition is optimal. We prove that the sorted phase is given by the condition

$$b > b_c := \sqrt{2} - 1, \tag{1.1}$$

see region III in Figure 1. Moreover, we show that the minimal discrepancy in this phase is of the order Mn. The region $b > \sqrt{2} - 1$ is precisely where Ferreira and Fontanari [5] observed that the corresponding statistical mechanical problem becomes "self-averaging."

[1] Note that the task of finding this partition is even easier than the task of sorting the n integers, which would take, on average, $\theta(n \log n)$ comparisons. Instead, the $(s + n)/2$ smallest integers can be found in strictly linear time in n.

Our analysis of the perfect and hard phases for $b < b_c$ is much more difficult. In this region, we use integral representations for the number of partitions with a given discrepancy and bias; these representations generalize those used in [2]. The asymptotic analysis of the resulting two-dimensional random integrals leads to saddle point equations for a saddle point described in terms two real parameters η and ζ. For discrepancies of order $o(M)$ (including, in particular, the case of perfect partitions), the saddle point equations determining ζ and η are:

$$\int_0^1 x \tanh(\zeta x + \eta)\, dx = 0, \quad \int_0^1 \tanh(\zeta x + \eta)\, dx = -b. \tag{1.2}$$

The solution (ζ, η) of these equations can be used to define the two convex curves in Figure 1. To this end, let [2]

$$L(\zeta, \eta) := b\eta + \int_0^1 \log(2 \cosh(\zeta x + \eta))\, dx \tag{1.3}$$

$$\rho(\zeta, \eta) := 1 - \frac{\tanh(\zeta + \eta) - \tanh(\eta)}{2\zeta}. \tag{1.4}$$

For (ζ, η) a solution of (1.2), we then define

$$\kappa_-(b) := -\log_2 \rho(\zeta, \eta), \quad \kappa_c(b) := \tfrac{1}{\log 2} L(\zeta, \eta). \tag{1.5}$$

From bottom to top, the two convex curves joining $(0, b_c)$ and $(1, 0)$ in Figure 1 are then given by $\kappa = \kappa_-(b)$ and $\kappa = \kappa_c(b)$.

Our results prove that, in the region $\kappa < \kappa_-(b)$, with probability tending to one as n tends to infinity (or, more succinctly, with high probability, w.h.p.) there exist perfect partitions; see region I in Figure 1. Moreover the number of perfect partitions is about $2^{(\kappa_c - \kappa)n}$ in this "perfect phase." We also prove that w.h.p. there are no perfect partitions in the region $b < b_c$ and $\kappa > \kappa_c(b)$. As in the unconstrained problem, we call this the "hard phase." Our results leave open the question of what happens in the narrow region $\kappa_- < \kappa < \kappa_c$, and also whether the optimal partition is unique in the hard phase.

We are also able to prove that these phase transitions correspond to qualitative changes in the solution space of the associated linear programming problem (LPP). In the actual optimum partitioning problem, each integer is put in one subset or the other. The relaxed version is defined by allowing any fraction of each integer to be put in either of the two partitions. Using our theorems on the typical behavior of integer partitioning problem and some general properties of the LPP, we show the following. In the sorted phase, i.e. for $b > b_c = \sqrt{2} - 1$, w.h.p. the LPP has a unique solution given by the sorted partition itself. For

[2] It turns out the solutions of the saddle point equations 1.2 are just the stationary points of the function $L(\zeta, \eta)$

$b < b_c$, i.e. in the perfect and hard phases, w.h.p. the relaxed minimum discrepancy is zero, and the total number of optimal basis solutions is exponentially large, of order $2^{k_c(b)n+O_p(n^{1/2})}$. Finally, in the perfect and hard phases, we consider the fraction of these basis solutions whose integer-valued components form an optimal integer partition of the subproblem with the corresponding subset of the weights. We show that this fraction is exponentially small. Moreover, except for the crescent-shaped region between $\kappa = \kappa_-(b)$ and $\kappa = \kappa_c(b)$, we show that the fraction is strictly exponentially smaller in the hard phase than in the perfect phase. This fraction thus represents some measure of the algorithmic difficulty of the problem..

The outline of the paper is as follows. In the next section, we define the problem in detail, and precisely state our main results. In Section 3, we introduce our integral representation and show how it leads to the relevant saddle point equations. We also give a brief heuristic derivation of some of the phase boundaries. The complete proofs are quite involved, and are presented in the full paper version [1] of this extended abstract.

2 Statement of Main Results

Let $X_1, \ldots, X_n$ be n independent copies of a generic random variable which is distributed uniformly on $\{1, \ldots, M\}$. We are interested in the case when M grows exponentially with n, and define κ as the exponential rate, i.e.

$$\kappa = n^{-1} \log_2 M. \tag{2.1}$$

To avoid trivial counterexamples, we will always assume that κ stay bounded away from both 0 and ∞ as $n \to \infty$. We will use $\mathbb{P}$ and $\mathbb{E}$, with or without subindex n, to denote the probability measure and the expectation induced by $\mathbf{X} = (X_1, \ldots, X_n)$.

A partition of integers into two disjoint subsets is coded by an n-long binary sequence $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$, $\sigma_j \in \{-1, 1\}$; so the subsets are $\{j : \sigma_j = 1\}$ and $\{j : \sigma_j = -1\}$. Obviously $\boldsymbol{\sigma}$ and $-\boldsymbol{\sigma}$ are the codes of the same partition. Given a partition $\boldsymbol{\sigma}$, we define its *discrepancy* (or energy), $d(\mathbf{X}, \boldsymbol{\sigma})$, and *bias* (or magnetization), $s(\boldsymbol{\sigma})$, as

$$d(\mathbf{X}, \boldsymbol{\sigma}) = |\boldsymbol{\sigma} \cdot \mathbf{X}|, \text{ with } \boldsymbol{\sigma} \cdot \mathbf{X} = \sum_{j=1}^{n} \sigma_j X_j, \tag{2.2}$$

$$s(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \cdot \mathbf{e} = |\{j : \sigma_j = 1\}| - |\{j : \sigma_j = -1\}|. \tag{2.3}$$

Here $\mathbf{e}$ is the vector $(1, \ldots, 1)$. Clearly $s(\boldsymbol{\sigma})$ is an integer in $\{-n, \ldots, n\}$, so let $s \in \{-n, \dots, n\}$ and define the bias density

$$b = |s|/n \tag{2.4}$$

so that $b \in [0,1]$. Note that by symmetry it suffices to consider $s(\boldsymbol{\sigma}) \in \{0, \dots, n\}$, so we will often take a non-negative integer $s \in \{0, \dots, n\}$, in which case $s = bn$. We define an *optimum partition* as a partition $\boldsymbol{\sigma}$ that minimizes the discrepancy $d(\mathbf{X}, \boldsymbol{\sigma})$ among all the partitions with bias equal to s, and a *perfect partition* as a partition $\boldsymbol{\sigma}$ with $|d(\mathbf{X}, \boldsymbol{\sigma})| \leq 1$.

Theorems B, C and D below describe our main results on the phases labelled I, II, and III in Figure 1 in the introduction. In the statement of these theorems we will use the parameters $\zeta, \eta, \kappa_c(b)$ and $\kappa_-(b)$ defined in (1.2) – (1.5). Before getting to principal results, we must begin with an existence statement for the parameters ζ, η.

Theorem A *Let $b < b_c$, where $b_c = \sqrt{2} - 1$. Then the saddle point equations (1.2) have a unique solution $(\zeta, \eta) = (\zeta(b), \eta(b))$.*

Let

$$Z_n(\ell, s) = Z_n(\ell, s; \mathbf{X}) \tag{2.5}$$

denote the random number of partitions $\boldsymbol{\sigma}$ with $\boldsymbol{\sigma} \cdot \mathbf{X} = \ell$ and $\boldsymbol{\sigma} \cdot \mathbf{e} = s$. Since $s(\boldsymbol{\sigma})$ has the same parity as n, and $d(\mathbf{X}, \boldsymbol{\sigma})$ has the same parity as $\sum_{j=1}^n X_j$, we will only consider values of s which have the same parity as n, and values of ℓ which have the same parity as $\sum_{j=1}^n X_j$. In the theorems below, we will not state these restrictions explicitly.

Our central goal is to use the saddle point solution in order to bound the $Z_n(\ell, s)$ for various given values of ℓ and s. To formulate our results in a compact, yet unambiguous form, we use a shorthand $a_n < a$ ($a_n > a$, resp.) instead of $\limsup a_n < a$ ($\liminf a_n > a$, resp.), even when the n-dependence of a_n is only implicit, as in $\kappa = n^{-1} \log_2 M$ and $b = |s/n|$. We will also use the notation $f_n = O_p(g_n)$ and $f_n = o_p(h_n)$ if f_n/g_n is bounded in probability and f_n/h_n goes to zero in probability, respectively. Also, as is customary, we will say that an event happens with high probability (w.h.p.) if the probability of this event approaches 1 as $n \to \infty$. In all our statements n, M, s and ℓ will be integers with $n \geq 1$, $M \geq 1$ and $s \geq 0$. Our main results in the perfect phase are summarized in the next theorem and remark.

Theorem B *Let $\ell = o(Mn^{1/2}), b < b_c$ and $\kappa < \kappa_-(b)$. Then w.h.p. $Z_n(\ell, s) \geq 1$ and*

$$Z_n(\ell, s) = 2^{[\kappa_c(b) - \kappa]n} e^{S_n n^{1/2} + o(n^{1/2})}, \tag{2.6}$$

where S_n converges in probability to a Gaussian with mean zero and variance $\sigma^2 = Var(\log(2\cosh(\zeta U + \eta)))$, with U uniformly distributed on $[0,1]$. Consequently, w.h.p., there exist exponentially many perfect partitions, with $\ell = 0$ if $\sum_j X_j$ is even, and $|\ell| = 1$ if $\sum_j X_j$ is odd.

Remark 2.1 *Under the conditions of Theorem B, we actually prove a much more accurate estimate. Namely, we show that there are* 2×2 *positive definite matrices* R *and* K *with deterministic entries, and a constant* $q<1$ *such that, with probability* $1-O(q^{\log^2 n})$,

$$Z_n(\ell,s)=\exp\left(\zeta\frac{\ell}{M}+\eta s+\sum_{j=1}^{n}\log(2\cosh(\zeta X_j/M+\eta))\right)$$
$$\times\frac{\exp(-\frac{1}{4}\boldsymbol{\tau}_n R^{-1}\boldsymbol{\tau}_n')}{\pi M n\sqrt{\det R}}(1+o(1)). \quad (2.7)$$

Here $\boldsymbol{\tau}_n$ *is a two-dimensional random vector which converges in probability to a Gaussian vector* $\boldsymbol{\tau}$ *with zero mean and covariance matrix* K*. See Section 5 of the full paper [1].*

The above expression for $Z_n(\ell,s)$ is much more complicated than its analogue in the unconstrained case, see equation (2.6) in [2]. Both the sum in the first exponent and the entire second exponent represent fluctuations which were not present in the unconstrained case, and which make the analysis of the perfect phase much more difficult here.

Our next theorem, which describes our main results on the hard phase, has two parts: The first shows that there are no perfect partitions above $\kappa=\kappa_c(b)$, and the second gives a bound on the number of optimum partition for $\kappa>\kappa_-$. To state the theorem, let $d_{opt}=d_{opt}(n;s)$ denote the discrepancy of the optimal partition, and let $Z_{opt}=Z_{opt}(n;s)$ denote the number of optimal partitions.

Theorem C *Let* $b<b_c$.

a) *If* $\kappa>\kappa_c(b)$*, then there exists a* $\delta>0$ *such that with probability* $1-O(e^{-\delta\log^2 n})$ *there are no perfect partitions, and moreover*

$$d_{opt}\geq 2^{n[\kappa-\kappa_c(b)]-O_p(n^{1/2})}. \quad (2.8)$$

b) *If* $\kappa>\kappa_-(b)$ *and* $\varepsilon>0$*, then there exists a constant* $\delta>0$ *such that*

$$d_{opt}\leq 2^{n[\kappa-\kappa_-(b)+\varepsilon]} \quad \textit{and} \quad Z_{opt}\leq 2^{n[\kappa_c(b)-\kappa_-(b)+\varepsilon]}, \quad (2.9)$$

both with probability $1-O(e^{-\delta\log^2 n})$.

Remark 2.2 *We believe that the bound in* (2.8) *is actually sharp. If we* assume *that this is the case, in fact, even if we assume that the weaker bound*

$$d_{opt}=2^{n(\kappa-\kappa_c+o_p(1))} \quad (2.10)$$

holds w.h.p. whenever $\kappa>\kappa_c$*, then we can significantly improve the second bound* (2.9)*. Indeed, under the assumption* (2.10)*,* Z_{opt} *grows subexponentially with* n *whenever* $\kappa>\kappa_c(b)$.

The optimum partition problem is much simpler for $b > b_c$. Our main result on the sorted phase is the following theorem.

Theorem D *Let $b > b_c$. Then w.h.p. the optimal partition is uniquely obtained by putting $(s+n)/2$ smallest integers X_j in one part, and the remaining $(n-s)/2$ integers into another part. W.h.p., d_{opt} is asymptotic to $\frac{Mn}{4}\Big[(1+b)^2 - 2\Big]$, i.e., of order Mn.*

By this theorem, for b sufficiently large, the partition is determined by the decreasing order of weights X_j, but not by the actual values of X_j.

It is a rather common idea to approximate an optimization problem defined with integer-valued variables by its relaxed version, where the variables are now allowed to assume any value within the real intervals whose endpoints are the admissible values of the original integer variables. In our case, the relaxed version is a linear programming problem (LPP) which can be stated as follows. Find the minimum value d_{opt} of d, subject to linear constraints

$$-d \le \sum_j \sigma_j X_j, \quad \sum_j \sigma_j X_j \le d, \qquad \sum_j \sigma_j = s, \quad -1 \le \sigma_j \le 1, \quad (1 \le j \le n). \tag{2.11}$$

As usual, the LPP has at least one basis solution, i.e. a solution $(\boldsymbol{\sigma}, d_{opt})$, which is an extreme (vertex) point of the polyhedron defined by the constraints (2.11). Let $N(\boldsymbol{\sigma}) := |\{j : \sigma_j \in (-1,1)\}|$ be the number of components of $\boldsymbol{\sigma}$ which are non-integer. It is easy for the reader to verify that $N(\boldsymbol{\sigma})$ is either 0 or 2 for all basis solutions $\boldsymbol{\sigma}$.

Our next theorem shows that the LPP inherits the phase diagram of the optimum partition problem, and moreover provides a limited way to quantify the relative algorithmic difficult of the optimal partition problem in the three regions. For $b > b_c$ the solutions of the initial partition problem and of its LPP version coincide. For $b < b_c$ they are very far apart, in terms of the *ratio* of respective optimal discrepancies. To state this precisely, we define $F_n(\kappa, b)$ to be the fraction of basis solutions $\boldsymbol{\sigma}$ with the property that the deletion of the $N(\boldsymbol{\sigma})$ components of $\boldsymbol{\sigma}$ with values in $(-1,1)$ produces an optimal integer partition for the corresponding subproblem with weights X_i. Henceforth, we will call this the "optimal subpartition property."

Theorem E *a) If $b > b_c$, then w.h.p. the sorted partition is a unique solution of the LPP, and thus $d_{opt}^{LPP} = \Theta(Mn)$ and $F_n(\kappa, b) = 1$.*

Let $b < b_c$.

b) Then w.h.p. $d_{opt}^{LPP} = 0$. In addition, w.h.p. there are $2^{[\kappa_c(b)+o(1)]n}$ basis solutions, each having either none or exactly two components $\sigma_i \ne \pm 1$.

c) W.h.p. $F_n(\kappa, b) = 2^{-[\kappa+o(1)]n}$ for $\kappa < \kappa_-(b)$, and $2^{-[\kappa_c(b)+o(1)]n} \le F_n(\kappa, b) \le 2^{-[\kappa_-(b)+o(1)]n}$ for $\kappa > \kappa_-(b)$.

Remark 2.3 *(i) If one assume that the number of optimal partitions Z_{opt} in the hard phase grows subexponentially with probability at least $1 - o(n^{-2})$ (see Remark 2.2 for a motivation of this assumption), our upper bound on the fraction $F_n(\kappa, b)$ in the hard phase can be improved to match the lower bound, yielding $F_n(\kappa, b) = 2^{-n[\kappa_c(b)+o(1)]}$ in the hard phase.*

(ii) If, on the other hand, the asymptotics of Remark 2.1 hold up to κ_c, more precisely, if one assumes that for $b < b_c$ and $\kappa < \kappa_c(b)$

$$Z_n(\ell, s) = 2^{n[\kappa_c(b)-\kappa+o(1))]} \tag{2.12}$$

holds with probability least $1 - o(n^{-2})$, then a bound of the form $F_n(\kappa, b) = 2^{-n[\kappa+o(1)]}$ can be extended to all $\kappa < \kappa_c$.

We close this section with an additional theorem on the expected number of perfect partitions. Consider the statements of Theorem C. Here again the situation is much more complicated than in the unconstrained case. By Theorem B and the lower bound in Theorem C(a), the minimum discrepancy changes from being at most one to being exponentially large as κ crosses the interval $[\kappa_-, \kappa_c]$. However, we can also prove that the *expected* number of perfect partitions remains exponentially large until κ reaches a value strictly exceeding κ_c. This is the content of the following theorem and remark.

Theorem F *Let $\ell \in \{-1, 0, 1\}$ and $b \in (0, 1)$. Then*

$$\lim_{n\to\infty} \Big[n^{-1} \log \mathbb{E}(Z_n(\ell, s)) - R(\kappa, b) \Big] = 0 \tag{2.13}$$

where

$$R(\kappa, b) = H((1 + b)/2) + \lambda b + \log(\lambda^{-1} \sinh \lambda) - \kappa \log 2, \tag{2.14}$$

with $H(u) = u \log(1/u) + (1 - u) \log(1/(1 - u))$, and λ satisfying $\coth \lambda = \lambda^{-1} - b$.

Remark 2.4 *Graphing the curve $R(\kappa, b) = 0$, i.e.*

$$\kappa = \kappa_e(b) := \frac{H((1 + b)/2) + \lambda b + \log(\lambda^{-1} \sinh \lambda)}{\log 2}, \tag{2.15}$$

we see that it lies strictly above $\kappa = \kappa_c(b)$, except at the only common point $\kappa = 1, b = 0$. In particular, the curve intersects the b-axis at $b = 0.56\cdots > b_c = 0.41\ldots$. Thus for the points (κ, b) between the curves $\kappa = \kappa_c(b)$ and $\kappa = \kappa_e(b)$, the expected number of perfect partitions grows exponentially, while w.h.p. there are no perfect partitions at all. This complex behavior did not manifest itself in the unconstrained optimum partitioning problem [2].

3 Outline of Proof Strategy

In this section, we define our notation, give the heuristics of the proof, and point out why naive extensions of the unconstrained analysis of [2] fail in the constrained case. The complete proofs are quite involved, and will be given in [1].

3.1 Sorted Partitions

We first discuss our strategy to prove that in region III, the optimal partition is sorted and has discrepancy of order Mn. To this end, we consider n weights $X_1, \ldots, X_n$, chosen uniformly at random from $\{1, \ldots, M\}$, and reorder them in such a way that their sizes are increasing, $X_{\pi(1)} \leq X_{\pi(2)} \leq \cdots \leq X_{\pi(n)}$, where $\pi(1), \ldots, \pi(n)$ is a suitable permutation of $1, \ldots, n$. Since M is assumed to grow exponentially with n, we have, in particular, $n^2 = o(M)$, which implies that w.h.p. no two weights are equal. So w.h.p. the permutation π is unique and $X_{\pi(1)} < X_{\pi(2)} < \cdots < X_{\pi(n)}$.

Given a bias $s > 0$, (with $s \equiv n (\text{mod } 2)$), we need to find an optimum partition that puts $k = (s+n)/2$ integers in one part, and the remaining $n-k$ integers into another part. One such feasible partition is obtained if we select the k smallest integers for the first part; we call it the sorted partition. It is coded by the $\boldsymbol{\sigma}$, with $\sigma_{\pi(i)} = 1$ for $i \leq k$ and $\sigma_{\pi(i)} = -1$ for $i > k$. If the total weight of $(n-k)$ largest weights is, at most, the total weight of k smallest weights, then it is intuitively clear that the sorted partition is optimal. More precisely: if

$$\delta_s(\boldsymbol{X}) = \sum_{j=1}^{k} X_{\pi(j)} - \sum_{j=k+1}^{n} X_{\pi(j)} \geq 0 \tag{3.1}$$

then the sorted partition is the unique, optimal partition, [3] and the minimal discrepancy is

$$d_{opt} = \delta_s(\boldsymbol{X}). \tag{3.2}$$

See Section 6 of [1] for a formal proof.

To determine the phase boundary of the phase III, we thus have to determine the region of the phase diagram in which w.h.p. the sorted partition meets the condition (3.1). Leaving the probabilistic technicalities out of our heuristic discussion, let us replace the condition (3.1) by its mean version,

[3] If $\delta_s(\boldsymbol{X}) = -1$, the sorted partition is still optimal (it is, in fact, perfect). But in general, it is not the unique optimum partition.

namely $\mathbb{E}(\delta_s(\mathbf{X})) \geq 0$. Consider an arbitrary $b \in (0,1]$. Let $x_0 = (1+b)/2$ and $M_0 = \lfloor x_0 M \rfloor$. For a typical set of weights $X_1, \ldots, X_n$, let us consider the sorted partition with $\sigma_j = 1$ for $X_j \leq M_0$, and $\sigma_j = -1$ for $X_j > M_0$. Since the probability that $X_j \leq M_0$ is equal to $\tilde{x}_0 = M_0/M = x_0 + O(M^{-1})$, we get that the expected number of weights X_j with $X_j \leq M_0$ is $n\tilde{x}_0$, implying that the expected bias is $2n\tilde{x}_0 - n = nb + O(n/M)$. The expected discrepancy can be calculated in a similar manner, giving the expression

$$\begin{aligned}
&\mathbb{E}\Big[\sum_j X_j \mathbb{I}\Big(X_j \leq \lfloor x_0 M \rfloor\Big) - \sum_j X_j \mathbb{I}\Big(X_j > \lfloor x_0 M \rfloor\Big)\Big] \\
&\quad = \frac{n}{M}\Big[M_0(1+M_0) - \frac{M(1+M)}{2}\Big] = \Big[x_0^2 - \frac{1}{2} + O(M^{-1})\Big] Mn \\
&\quad = \Big[\Big(\frac{b+1}{2}\Big)^2 - \frac{1}{2} + O(M^{-1})\Big] Mn
\end{aligned} \tag{3.3}$$

So, $\mathbb{E}(\delta_s(\mathbf{X}))$ is large positive, of order Mn, iff $(b+1)^2/4 - 1/2 > 0$, or equivalently $b > b_c = \sqrt{2} - 1$. In Section 6 of [1], we prove the condition $b > b_c$ is both necessary and sufficient for $\delta_s(\mathbf{X})$ to be, w.h.p., positive, of order Mn. In language of statistical mechanics, we show that, for $b > b_c$, $\delta_s(\mathbf{X})$ is "self-averaging," i.e., its distribution is sharply concentrated around $\mathbb{E}(\delta_s(\mathbf{X}))$.

Remark 3.1 *On the heuristic level presented here, the above arguments can easily be generalized to an arbitrary distribution for the weights $X_1, \ldots, X_n$, as long as these weights are independent copies of a generic (discrete) variable X with a reasonably well behaved probability distribution. Assuming, e.g., that the variable X/M has a limiting distribution with density μ, one obtains that the critical value of b is given by $b_c = b_c(\mu) = 2\int_0^{x_0} \mu(x)\, dx - 1$, where x_0 is determined by the equation $\int_0^{x_0} x\mu(x)\, dx = \int_{x_0}^{\infty} x\mu(x)\, dx$. However, we have not tried to extend all our results to this more general μ-density case.*

3.2 Integral Representations

Let us now turn to the much more difficult region $b < b_c$. Without loss of generality, we may take $s \geq 0$, so that $b = s/n$.

Let $Z_n(\ell, s) = Z_n(\ell, s; \mathbf{X})$ denote the total number of partitions $\boldsymbol{\sigma}$ such that $\boldsymbol{\sigma} \cdot \mathbf{X} = \ell$ and $\boldsymbol{\sigma} \cdot \mathbf{e} = s$. Guided by the results of [2], one might hope to prove that, as the parameter $\kappa = n^{-1} \log_2 M$ is varied, the model undergoes a phase transition between a region with exponentially many perfect partitions and a region with no perfect partitions. Since perfect partitions correspond to $\ell = 0$ or $\ell = \pm 1$, we will be mainly interested in $Z_n(\ell, s)$ for $|\ell| \leq 1$, while s will typically be chosen proportional to n.

A starting point in [2] was an integral (Fourier-inversion) type formula for $Z_n(\ell) = Z_n(\ell; \mathbf{X})$, the total number of $\boldsymbol{\sigma}$'s such that $\boldsymbol{\sigma} \cdot \mathbf{X} = \ell$, namely

$$Z_n(\ell) = \frac{2^n}{\pi} \int\limits_{x \in (-\pi/2, \pi/2]} \cos(\ell x) \prod_{j=1}^{n} \cos(x X_j)\, dx. \tag{3.4}$$

We need to derive a two-dimensional counterpart of that formula for $Z_n(\ell, s)$. To this end, let us first recall that by (2.3), $s = 2|\{j : \sigma_j = 1\}| - n$, so that a generic value s of $s(\boldsymbol{\sigma})$ must meet the condition $n+s \equiv 0 (\text{mod } 2)$. In a similar way, we get that $\boldsymbol{\sigma} \cdot \mathbf{X}$ has the same parity as the sum $\sum_j X_j$. Keeping this in mind, we have that on the event $\{\sum_j X_j \equiv \ell (\text{mod } 2)\}$, for $n + s \equiv 0 (\text{mod } 2)$,

$$\mathbb{I}(\boldsymbol{\sigma} \cdot \mathbf{X} = \ell,\, \boldsymbol{\sigma} \cdot \mathbf{e} = s) = \frac{1}{\pi^2} \iint\limits_{x,y \in (-\pi/2, \pi/2]} e^{i(\boldsymbol{\sigma} \cdot \mathbf{X} - \ell)x}\, e^{i(\boldsymbol{\sigma} \cdot \mathbf{e} - s)y}\, dxdy, \tag{3.5}$$

thus extending (4.6) in [2]. Multiplying both sides of the identity by 2^n, and summing over all $\boldsymbol{\sigma}$, we obtain

$$\begin{aligned} Z_n(\ell, s) &= \frac{2^n}{\pi^2} \iint\limits_{x,y \in (-\pi/2, \pi/2]} e^{-i(\ell x + s y)} \prod_{j=1}^{n} \cos\Big(x X_j + y\Big)\, dxdy \\ &= 2^n\, \mathbb{P}_{1/2}\Big(\boldsymbol{\sigma} \cdot \mathbf{X} = \ell,\, \boldsymbol{\sigma} \cdot \mathbf{e} = s \Big| \boldsymbol{X}\Big), \end{aligned} \tag{3.6}$$

where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ is a sequence of i.i.d. Bernoulli random variables with probability of $\sigma_i = \pm 1$ equal to $1/2$.

We would like to estimate the asymptotics of the integral in (3.6), which is equivalent to proving a local limit theorem for the conditional probability in (3.6). In general, to compute—via local limit theorems—the probability that some random variable A takes the value a, it must be the case that the corresponding expectation of A is near a. Thus the analogue of the representation (3.6) for the unconstrained problem was well adapted to the analysis of perfect partitions. Indeed, in that case, we wanted to estimate $\mathbb{P}_{1/2}(|\boldsymbol{\sigma} \cdot \mathbf{X}| \leq 1 | \mathbf{X})$, and we had $\mathbb{E}_{1/2}(\boldsymbol{\sigma} \cdot \mathbf{X} | \mathbf{X}) = 0$. However, in the constrained case, this strategy cannot be expected to work for $b > 0$, since $s = bn$ is very far from the expectation of $\boldsymbol{\sigma} \cdot \mathbf{e}$, namely $\mathbb{E}_{1/2}(\boldsymbol{\sigma} \cdot \mathbf{e} | \mathbf{X}) = 0$.

To resolve this substantial difficulty, we introduce a *two-parameter* family of distributions for σ_j as follows: Given $\xi, \eta \in \mathbb{R}$, let $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_n)$ be a sequence of random variables such that, conditioned on $\mathbf{X}$, $\sigma_1, \ldots, \sigma_n$ are mutually independent, and

$$\mathbb{P}(\sigma_j = 1 | \mathbf{X}) = P(\xi X_j + \eta), \qquad \mathbb{P}(\sigma_j = -1 | \mathbf{X}) = 1 - P(\xi X_j + \eta), \tag{3.7}$$

where

$$P(u) := \frac{e^{-u}}{2 \cosh u}. \tag{3.8}$$

In terms of these random variables, $Z_n(\ell, s)$ can be rewritten as

$$\begin{aligned} Z_n(\ell,s) &= e^{nL_n(\xi,\eta;\mathbf{X})}\,\mathbb{P}\Big(\boldsymbol{\sigma}\cdot\mathbf{X}=\ell,\ \boldsymbol{\sigma}\cdot\mathbf{e}=s|\mathbf{X}\Big) \\ &= e^{nL_n(\xi,\eta;\mathbf{X})}\,\frac{1}{\pi^2}\iint\limits_{x,y\in(-\pi/2,\pi/2]} e^{-i(\ell x+sy)}\mathbb{E}\Big(\exp(i(x\boldsymbol{\sigma}\cdot\mathbf{X}+y\boldsymbol{\sigma}\cdot\mathbf{e}))|\mathbf{X}\Big)\,dxdy, \end{aligned} \tag{3.9}$$

where

$$L_n(\xi,\eta;\mathbf{X}) := \frac{\ell\xi}{n}+\frac{s\eta}{n}+\frac{1}{n}\sum_{j=1}^{n}\log(2\cosh(\xi X_j+\eta)). \tag{3.10}$$

Indeed, fix $\xi, \eta \in \mathbb{R}$. Then $Z_n(\ell, s)$ can be rewritten as

$$\begin{aligned} Z_n(\ell,s) &= \sum_{\boldsymbol{\tau}\in\{-1,+1\}^n}\mathbb{I}(\boldsymbol{\tau}\cdot\mathbf{X}=\ell,\ \boldsymbol{\tau}\cdot\mathbf{e}=s) \\ &= \sum_{\substack{\boldsymbol{\tau}:\boldsymbol{\tau}\cdot\mathbf{X}=\ell,\\ \boldsymbol{\tau}\cdot\mathbf{e}=s}} e^{\xi(\ell-\boldsymbol{\tau}\cdot\mathbf{X})+\eta(s-\boldsymbol{\tau}\cdot\mathbf{e})} = e^{\xi\ell+\eta s}\sum_{\substack{\boldsymbol{\tau}:\boldsymbol{\tau}\cdot\mathbf{X}=\ell,\\ \boldsymbol{\tau}\cdot\mathbf{e}=s}}\prod_{j=1}^{n}e^{-(\xi X_j+\eta)\tau_j} \\ &= \left[e^{\xi\ell+\eta s}\prod_{j=1}^{n}(2\cosh(\xi X_j+\eta))\right]\sum_{\substack{\boldsymbol{\tau}:\boldsymbol{\tau}\cdot\mathbf{X}=\ell,\\ \boldsymbol{\tau}\cdot\mathbf{e}=s}}\prod_{j'=1}^{n}P\Big((\xi X_{j'}+\eta)\tau_{j'}\Big) \qquad (3.11) \\ &= e^{nL_n(\xi,\eta;\mathbf{X})}\sum_{\substack{\tau:\tau\cdot\mathbf{X}=\ell,\\ \tau\cdot\mathbf{e}=s}}\prod_{j=1}^{n}\mathbb{P}(\sigma_j=\tau_j|\mathbf{X}) \\ &= e^{nL_n(\xi,\eta;\mathbf{X})}\mathbb{P}(\boldsymbol{\sigma}\cdot\mathbf{X}=\ell,\boldsymbol{\sigma}\cdot\mathbf{e}=s|\mathbf{X}), \end{aligned}$$

since $P(-u) = 1 - P(u)$, see equation (3.8).

3.3 Saddle Point Equations and their Solution

Given ξ, η, we now face the problem of determining an asymptotic value of the *local* probability in (3.9). This will obviously be easier if the chosen parameters ℓ and s are among the more likely values of $\boldsymbol{\sigma}\cdot\mathbf{X}$ and $\boldsymbol{\sigma}\cdot\mathbf{e}$, respectively. A natural choice is to take ℓ and s equal to their expected values, that is

$$\mathbb{E}\Big(\sigma\cdot\mathbf{X}|\mathbf{X}\Big)=\ell,\quad \mathbb{E}\Big(\boldsymbol{\sigma}\cdot\mathbf{e}|\mathbf{X}\Big)=s, \tag{3.12}$$

or explicitly (using (3.8), (3.7))

$$\sum_{j=1}^{n}X_j\tanh(\xi X_j+\eta)=-\ell,\qquad \sum_{j=1}^{n}\tanh(\xi X_j+\eta)=-s. \tag{3.13}$$

Note that the equations (3.13) also arise naturally in an apparently different approach to estimate the integral in (3.6), the "method of steepest descent." In our context, this corresponds to a complex shift of the integration path, i.e., to changing the path of integration for x to the complex path from $-\pi/2 + i\xi$ to $-\pi/2 + i\xi$, and the path of integration for y to the complex path from $-\pi/2 + i\eta$ to $-\pi/2 + i\eta$, where ξ and η are determined by a suitable saddle point condition. For general ξ and η, this leads to (3.9), while the saddle point conditions turn our to be nothing but (3.13). In fact, this is how we first obtained (3.9) and (3.13).

Both approaches raise the question of uniqueness and existence of a solution to the saddle point equations (3.13). In this context, it is useful to realize that the conditions (3.13) can be rewritten as

$$\frac{\partial L_n(\xi,\eta;\mathbf{X})}{\partial \xi} = 0, \quad \frac{\partial L_n(\xi,\eta;\mathbf{X})}{\partial \eta} = 0. \tag{3.14}$$

Therefore any solution (ξ, η) is a stationary point of the strictly convex function $L_n(\xi, \eta; \mathbf{X})$. If a solution exists, it is therefore the unique *minimum* point of L_n. Using the first equation in (3.9), we see also that (ξ, η) *maximizes* the local probability $\mathbb{P}(\boldsymbol{\sigma} \cdot \mathbf{X} = \ell, \boldsymbol{\sigma} \cdot e = s|\mathbf{X})$, and hence makes it easier to do an asymptotic analysis. This observation justifies our choice of ξ, η.

In the actual proof, we modify this approach a little since the solution $\xi = \xi(\mathbf{X})$, $\eta = \eta(\mathbf{X})$ does not lend itself to a rigorous analysis of $\mathbb{P}(\boldsymbol{\sigma} \cdot \mathbf{X} = \ell, \boldsymbol{\sigma} \cdot e = s|\mathbf{X})$. Instead, we will resort to "suboptimal" $\xi = \zeta/M$, η, where ζ, η are nonrandom constants, and (ζ, η) is a solution of nonrandom equations, obtained by replacing the (scaled) sums in (3.13) with their weak-law limits, see equations (3.18) below. This way we will be able to establish an explicit asymptotic formula for $Z_n(\ell, s)$, which will ultimately lead us to determine the phase boundaries.

In Section 4 of [1], we will show that these deterministic equations have a (unique) solution $\zeta = \zeta(b)$, $\eta = \eta(b)$ iff $b < b_c = \sqrt{2} - 1$, the same b_c that determines the sorted phase. In other words, the threshold b_c plays two seemingly unrelated roles: both as a threshold value of b for solvability of the deterministic saddle point equations (3.18), and as a threshold for the sorted partition being optimal. On an informal level, the reason for the coincidence is as follows: For simplicity, suppose that the weights X_j are all distinct, so that $X_1 < \cdots < X_n$ after reordering. As b approaches the point where the solutions (ζ, η) to the saddle point equations (3.18) stop existing, these solutions actually diverge, one tending to $+\infty$ and the other to $-\infty$. According to equations (3.7) and (3.8), this in turn means that $\mathbb{P}(\sigma_j = 1|\mathbf{X})$ tends to zero or one, depending on whether $j < j_o$ or $j > j_o$, where $j_o = |\{j : \sigma_j = -1\}|$ is the cutoff of the sorted partition for $\mathbf{X}$ with bias $s = nb_c$. Hence, the product measure $\mathbb{P}(\boldsymbol{\sigma} \cdot \mathbf{X} = \ell, \boldsymbol{\sigma} \cdot \mathbf{e} = s|\mathbf{X})$ tends to a delta function on the (unique) sorted partition which is the solution to the number partitioning problem for $\mathbf{X}$ at $b = b_c$. See Subsection 7.1 of [1] for details.

3.4 Asymptotic behavior of $Z_n(\ell, s)$.

Proceeding with our heuristic discussion, let us simply assume that the equations (3.13) do have a solution $\xi = \xi(\mathbf{X}), \eta = \eta(\mathbf{X})$. Then we may hope that, with this choice of the parameters $\xi = \xi(\mathbf{X})$, $\eta = \eta(\mathbf{X})$, we have a reasonable chance to prove—at least for the likely values of $\mathbf{X}$—a local limit theorem for the *conditional* probability in (3.9), namely that w.h.p.

$$\mathbb{P}\Big(\boldsymbol{\sigma}\cdot\mathbf{X} = \ell, \boldsymbol{\sigma}\cdot\mathbf{e} = s|\mathbf{X}\Big) \sim \frac{2}{\pi\sqrt{\det\, Q}}, \tag{3.15}$$

where

$$Q = \begin{pmatrix} \mathrm{Var}(\boldsymbol{\sigma}\cdot X) & \mathrm{cov}(\boldsymbol{\sigma}\cdot\mathbf{X}, \boldsymbol{\sigma}\cdot\mathbf{e}) \\ \mathrm{cov}(\boldsymbol{\sigma}\cdot\mathbf{X}, \boldsymbol{\sigma}\cdot\mathbf{e}) & \mathrm{Var}(\boldsymbol{\sigma}\cdot\mathbf{e}) \end{pmatrix}. \tag{3.16}$$

Here the (co)variances are conditioned on $\mathbf{X}$, so, e.g., $Q_{11} = \mathrm{Var}(\boldsymbol{\sigma}\cdot\mathbf{X}|\mathbf{X})$. If (3.15) holds then by (3.11), w.h.p.,

$$Z_n(\ell, s) \sim e^{nL_n(\xi,\eta;\mathbf{X})}\,\frac{2}{\pi\sqrt{\det\, Q}} = e^{nL_n(\xi,\eta;\mathbf{X})}\,\frac{2}{\pi n M\sqrt{\det\, R^{(n)}}}, \tag{3.17}$$

where $R^{(n)}$ is the matrix with matrix elements $R^{(n)}_{11} = \frac{1}{nM^2}\mathrm{Var}(\boldsymbol{\sigma}\cdot X)$, $R^{(n)}_{12} = R^{(n)}_{21} = \frac{1}{nM}\mathrm{cov}(\boldsymbol{\sigma}\cdot\mathbf{X}, \boldsymbol{\sigma}\cdot\mathbf{e})$ and $R^{(n)}_{22} = \frac{1}{n}\mathrm{Var}(\boldsymbol{\sigma}\cdot\mathbf{e})$.

Note that, in the limit $M \to \infty$, X_j/M are independent, uniform random variables in $[0, 1]$. We therefore expect that as $M, n \to \infty$ with $\kappa = n^{-1}\log_2 M$ fixed, both $\zeta(\mathbf{X}) := M\xi(\mathbf{X})$ and $\eta(\mathbf{X})$ are close, in probability, to the deterministic ζ, η, defined as the roots of the averaged version of the "saddle point equations" (3.13), namely

$$\int_0^1 x\tanh(\zeta x + \eta)\,dx = -\frac{\ell}{Mn}, \quad \int_0^1 \tanh(\zeta x + \eta)\,dx = -b, \quad b = \frac{s}{n}. \tag{3.18}$$

Recall that, without loss of generality, we have taken $s \geq 0$, so $b \geq 0$.

Furthermore, approximating $\xi(\mathbf{X})$ and $\eta(\mathbf{X})$ by $M\zeta$ and η, respectively and using the bound $|d\cosh u/du| \leq 1$, it is easy to see that, because of the weak law of large numbers, w.h.p.

$$\begin{aligned} L_n(\xi(\mathbf{X}), \eta(\mathbf{X}); \mathbf{X}) &= \frac{1}{n}\sum_{j=1}^{n}\log\Big(e^{\ell\xi(\mathbf{X})+s\eta(\mathbf{X})}2\cosh(\xi(\mathbf{X})X_j + \eta(\mathbf{X}))\Big) \\ &\sim \frac{\ell}{Mn}\zeta + b\eta + \int_0^1 \log(2\cosh(\zeta x + \eta))\,dx, \end{aligned} \tag{3.19}$$

and similarly for the matrix elements of $R^{(n)}$,

$$R_{ij}^{(n)} \sim \int_0^1 x^{2-(i+j)}(1-\tanh^2(\zeta x+\eta))\,dx. \tag{3.20}$$

Putting everything together, we thus may hope to prove that for $|\ell| \leq 1$ and M growing exponentially with n, (i.e. $\log_2 M \sim \kappa n$ for some n-independent κ), we have w.h.p.

$$\frac{1}{n}\log Z_n(\ell,s) \sim \int_0^1 \log(2\cosh(\zeta x+\eta))\,dx + b\eta - \kappa = \kappa_c(b) - \kappa, \tag{3.21}$$

suggesting that for $\kappa < \kappa_c(b)$ there are exponentially many perfect partitions, while for $\kappa > \kappa_c(b)$ there are none.

However, this informal argument is too naive. Equation (3.21) could not possibly hold for $\kappa > \kappa_c(b)$. Indeed, $Z_n(\ell, s)$ is an integer, and thus cannot be asymptotically equivalent to an exponentially small, yet positive number. This means that a rigorous proof of (3.21) must be based on the condition $\kappa < \kappa_c(b)$. But our heuristic discussion provides no clue as to how this condition might enter the picture. Furthermore, our attempts to find such a proof are stymied by mutual dependence of the random variables $\mathbb{P}(\sigma_j = 1|\mathbf{X})$, $(1 \leq j \leq n)$, a consequence of the fact that $(\xi(\mathbf{X}), \eta(\mathbf{X}))$ depends, in an unwieldy manner, on the whole $\mathbf{X}$. This complicated dependence of $(\xi(\mathbf{X}), \eta(\mathbf{X}))$ on $\mathbf{X}$ would have made it very hard to gain an insight into the random fluctuations of the sum in (3.19), even if we had found a proof.

Fortunately, once we have informally connected $(\xi(\mathbf{X}), \eta(\mathbf{X}))$ to (ζ, η) via $\xi(\mathbf{X}) = (1 + o_p(1))\zeta/M$, $\eta(\mathbf{X}) = (1 + o_p(1))\eta$, we may try to use the *sub-optimal* parameters $(\zeta/M, \eta)$ instead. The corresponding random variables $\mathbb{P}(\sigma_j = 1|\mathbf{X})$ each depend on their own X_j, and are thus mutually independent. A key technical issue is whether the suboptimal parameters are good enough to get an asymptotic formula for the corresponding probability $\mathbb{P}(\boldsymbol{\sigma} \cdot \mathbf{X}, \boldsymbol{\sigma} \cdot \mathbf{X} = s|\mathbf{X})$, given that now the random equations (3.13) may hold only approximately. Our proof in [1] shows that they are indeed sufficient. With those parameters, we will be able to get a sharp explicit approximation for $\log Z_n(\ell, s)$, at least in the range $\kappa < \kappa_-(b)$.

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